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Planar super-Landau models

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Abstract: In previous papers we solved the Landau problems, indexed by $2M$, for a particle on the "superflag" SU(2|1)/[U(1) × U(1)], the $M = 0$ case being equivalent to the Landau problem for a particle on the "supersphere" SU(2|1)/[U(1|1)]. Here we solve these models in the planar limit. For $M = 0$ we have a particle on the complex superplane $\mathbb{C}^{(1|1)}$; its Hilbert space is the tensor product of that of the Landau model with the 4-state space of a "fermionic" Landau model. Only the lowest level is ghost-free, but for $M > 0$ there are no ghosts in the first $[2M] + 1$ levels. When $2M$ is an integer, the $(2M + 1)$th level states form short supermultiplets as a consequence of a fermionic gauge invariance analogous to the "kappa-symmetry" of the superparticle.

Keywords: Integrable Equations in Physics, Superspaces.

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Contents

1. Introduction

In 1930 Landau posed and solved the quantum mechanical problem of a charged particle in a plane orthogonal to a uniform magnetic field, showing in particular that the particle's energy is restricted to a series of "Landau levels" [1]. In the low-energy limit only the lowest level is relevant, and the low-energy physics is described by a first-order "Lowest-Landau-Level" (LLL) model with a phase space that is a non-commutative version of the original configuration space. In more recent times, this connection with non-commutative geometry has led to a revival of interest in Landau-type models.

In 1983 Haldane generalized the Landau model to a particle on a sphere in $\mathbb{E}^3$ of radius R, in the uniform magnetic field B generated by a magnetic monopole at the centre of the sphere [2]. If the monopole strength is n times the minimal value allowed by the Dirac quantization condition then $B \propto n/R^2$ and the planar Landau model is recovered in the limit that $n \to \infty$ and $R \to \infty$ with B kept fixed. If instead one takes the limit as $R \to 0$ with n fixed then one finds a LLL model with an action that is n times the minimal U(1) Wess-Zumino (WZ) term associated with the description of the sphere as $\mathrm{SU}(2)/\mathrm{U}(1) \cong CP^1$. The phase space of this LLL model is a fuzzy sphere [3].

In two previous papers [4, 5] we have considered Landau models for a particle on a *superspace* with CP^1 body. The minimal dimension symmetric superspace with this property is $CP^{(1|1)} \cong \mathrm{SU}(2|1)/\mathrm{U}(1|1)$, which we called the *supersphere*.[1] The LLL model for a particle on the supersphere yields a physical realization of the fuzzy supersphere [4].

[1] Other definitions of "supersphere" occur in the literature (references can be found in our previous papers) but none is obviously equivalent to our definition.

Although this model is perfectly physical, the full Landau model for a particle on the supersphere is unphysical because the higher Landau levels all contain ghosts; i.e., states of negative norm. This feature is directly related to the presence of a non-canonical fermionic kinetic term with two time derivatives.

In an attempt to circumvent this problem, we considered in [5] the Landau model for a particle on the coset superspace $\mathrm{SU}(2|1)/[\mathrm{U}(1) \times \mathrm{U}(1)]$. This supermanifold again has CP^1 body but it is not a symmetric superspace; it is an analog of the flag manifold $\mathrm{SU}(3)/[\mathrm{U}(1) \times \mathrm{U}(1)]$, and for this reason we called it the "superflag". For given magnetic field strength there is a one-parameter family of superflag Landau models parametrized, in the notation of [5], by the coefficient $2M$ of an additional, purely "fermionic", WZ term. Although the relationship between the superflag and supersphere Landau models was not spelled out in our earlier work, one can show that supersphere model is equivalent to the $M = 0$ superflag model. The parameter M has no effect on the energy levels, which are therefore the same as those of the supersphere Landau model, but one now finds that states in the first $[2M] + 1$ levels have positive norm, although all higher levels still contain states of negative norm.[2] When $2M$ is an integer, the $(2M + 1)$th level states form a short representation of $\mathrm{SU}(2|1)$ as a consequence of the presence of zero-norm states.

One aim of this paper is to elucidate these features of spherical super-Landau models by an analysis of the much simpler models obtained in the planar limit. The planar limit of the supersphere is the complex superplane $\mathbb{C}^{(1|1)}$. This can be viewed as the coset superspace

$$\mathrm{IU}(1|1)/[\mathrm{U}(1|1) \times \mathcal{Z}]\,, \tag{1.1}$$

where $\mathrm{IU}(1,1)$ is a central extension of a contraction of $\mathrm{SU}(2|1)$ and $\mathcal{Z}$ is the abelian group generated by the central charge. The corresponding "superplane Landau model" has a quadratic Lagrangian and a Hilbert space that is the tensor product of the standard Landau model Hilbert space with a 4-state space of a "fermionic Landau model". Analysis of this 4-state system shows clearly how negative norm states arise as a consequence of the two-derivative, and hence non-canonical, fermion kinetic term, but also why the LLL is ghost-free.

This analysis of the superplane Landau model suggests a strategy for removing the negative norm states by modifying the Lagrangian in such a way as to cancel the two-derivative, or "second-order", fermion kinetic term. This requires the introduction of interactions with an additional complex "Goldstino" variable and the introduction of a first-order kinetic term for it, with coefficient $2M$. The resulting model, which is the Landau model for a particle on the coset superspace

$$\mathrm{IU}(1|1)/[\mathrm{U}(1) \times \mathrm{U}(1) \times \mathcal{Z}]\,, \tag{1.2}$$

is precisely the planar limit of the superflag Landau model; we call it the "planar superflag Landau model". The cancellation of the second-order fermion term in this "planar superflag" Landau model is incomplete, however, because it survives in a "bodyless" form

[2] $[2M]$ is the integer part of $2M$.

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with nilpotent Goldstino bilinear coefficient. At the quantum level, this results not in the elimination of all negative norm states but rather in their banishment to the higher Landau levels, exactly as found in [5] for the superflag Landau model. One may then discard these levels to arrive at a model with a finite-dimensional Hilbert space that generalizes the LLL model obtained by the truncation to the ground state level, exactly as argued in [5] for the superflag Landau model.

Thus many of the peculiar properties of the supersphere and superflag models of [4, 5] survive the planar limit and are readily understood in this simpler context. In particular, the structure of the phase-space constraints is simple to analyse in the planar limit, and it explains why zero norm states appear in the $(2M+1)$th level when $2M$ is an integer. Recall that the Hamiltonian formulation of models with canonical fermion kinetic terms requires fermionic constraints on the phase superspace. No such constraints are needed for the superplane model as it has non-canonical, second-order, fermion kinetic terms, but constraints are needed for the $(M>0)$ planar superflag model. Usually, these constraints are either all "second class" (in Dirac's terminology) or (as in many superparticle models) a definite mixture of first and second class, the first class constraints indicating the presence of a fermionic gauge invariance. Here we find fermionic constraints that are second class everywhere *except on a particular energy surface*, where they are of mixed type.[3] This implies a fermionic gauge invariance analogous to the "kappa-symmetry" of the superparticle, but restricted to a subspace of definite energy. Because of energy quantization, this has an effect on the quantum theory only when $2M$ is an integer, and it is responsible for the zero-norm states in the $(2M+1)$th level.

We shall begin with an analysis of the superplane Landau model. Its quantization is essentially trivial because the Lagrangian is quadratic, but it provides a useful starting point, and a simple context in which one can discuss the IU(1|1) symmetries. We then show how a modification of this Lagrangian to include interactions with a Goldstino variable yields the planar superflag Landau models, indexed by the coefficient $2M$ of a fermionic WZ term. The equivalence with the superplane Landau model for $M=0$ is then established; this equivalence is not obvious and requires careful consideration of the Hamiltonian constraint structure of the planar superflag models. We then use this Hamiltonian analysis to quantize the planar superflag model, using the methods of our previous papers. Finally, we present a geometrical formulation of our results.

2. The superplane Landau model

We begin with the superplane Landau model. By "superplane" we mean the superspace $\mathbb{C}^{(1|1)}$ parametrized by complex coordinates (z,ζ), where z is a complex number and ζ a complex anticommuting variable. The superplane Lagrangian is

$$L_0 = L_b + L_f\,, \tag{2.1}$$

[3] Something similar occurs for higher-dimensional Chern-Simons theories [6] but in the context of bosonic constraints.

where

$$L_b = |\dot{z}|^2 - i\kappa\left(\dot{z}\bar{z} - \dot{\bar{z}}z\right) \tag{2.2}$$

is the Lagrangian for the standard planar Landau model with energy spacing 2κ (which we take to be positive), and

$$L_f = \dot{\zeta}\dot{\bar{\zeta}} - i\kappa\left(\dot{\zeta}\bar{\zeta} + \dot{\bar{\zeta}}\zeta\right) \tag{2.3}$$

is the Lagrangian for a fermionic Landau model in terms of an anticommuting complex variable ζ. We call the total Lagrangian L_0 because it is quadratic; we will later add interaction terms to get the Lagrangian of the planar superflag Landau model.

The Hilbert space of this model is obviously a tensor product of the Hilbert space of the Landau model with that of the fermionic Landau model with Lagrangian L_f, so all the new features must arise from the latter model, which we therefore analyse first. Because L_f contains a "second-order" kinetic term, and second-order is "higher-order" for fermions, we should expect ghosts (negative norm states). We shall show that this intuition is indeed correct, but also that all LLL levels have positive norm. This too is expected since the LLL states are all that survive in large κ limit in which all terms of the second order in time derivative become irrelevant.

Having analysed the fermionic Landau model, the spectrum of states of the full superplane model, and their norms, is easily determined. However, the degeneracies in the spectrum are consequences of symmetries of the full Lagrangian. The relevant symmetry group is the supergroup IU(1|1) obtained by a contraction of the SU(2|1) symmetry of the supersphere. We exhibit these symmetries, and show precisely how IU(1|1) is obtained from SU(2|1).

2.1 Fermionic Landau model

For the purposes of comparison we first summarize Landau's results for the standard, "bosonic" Landau model. The equation of motion has the general solution

$$z = z_0 + (\dot{z}_0/\kappa)e^{-i\kappa t}\sin\kappa t\,, \tag{2.4}$$

so the motion is periodic with angular frequency 2κ. To pass to the quantum theory it is convenient to use the Hamiltonian form of the Lagrangian

$$L_b = \dot{z}p + \dot{\bar{z}}\bar{p} - H_b\,, \qquad H_b = |p + i\kappa\bar{z}|^2\,, \tag{2.5}$$

where p is the complex momentum conjugate to z. To obtain the quantum Hamiltonian $\hat{H}_b$ we then make the replacements

$$p \to \hat{p} = -i\partial_z\,, \qquad \bar{p} \to \hat{\bar{p}} = -i\partial_{\bar{z}}\,. \tag{2.6}$$

There is a trivial ordering ambiguity but the natural symmetric ordering yields

$$\hat{H}_b = a^\dagger a + \kappa\,, \tag{2.7}$$

where

$$a = i\left(\partial_{\bar{z}} + \kappa z\right)\,, \qquad a^\dagger = i\left(\partial_z - \kappa\bar{z}\right)\,. \tag{2.8}$$

These operators satisfy the creation and annihilation operator commutation relation

$$[a, a^\dagger] = 2\kappa\,. \tag{2.9}$$

The ground states, which span the LLL, have energy κ and are annihilated by a. States in the higher Landau levels are obtained by acting on a LLL state with $a^\dagger$, so the energy levels are $E = 2\kappa(N + 1/2)$ for non-negative integer N.

The equation of motion of the fermionic Landau model has the general solution

$$\zeta = \zeta_0 + (\dot{\zeta}_0/\kappa)e^{-i\kappa t}\sin\kappa t\,, \tag{2.10}$$

so the motion is again periodic with period 2κ. The Hamiltonian form of the Lagrangian is

$$L_f = -i\dot{\zeta}\pi - i\dot{\bar{\zeta}}\bar{\pi} - H_f\,, \qquad H_f = (\bar{\pi} - \kappa\zeta)\left(\pi - \kappa\bar{\zeta}\right)\,, \tag{2.11}$$

where π is the momentum conjugate to ζ. We use here the Grassmann-odd phase space conventions of [7] for which $\bar{\pi}$ is the complex conjugate of π. Note that this Lagrangian is invariant under the rotational and translational isometries of the complex Grassmann plane (together with a corresponding phase rotation of π). To pass to the quantum theory we make the replacements

$$\pi \to \hat{\pi} = \partial_\zeta\,, \qquad \bar{\pi} \to \hat{\bar{\pi}} = \partial_{\bar{\zeta}}\,, \tag{2.12}$$

where the Grassmann-odd derivatives should be understood as left-derivatives. There is a trivial ordering ambiguity in the Hamiltonian, but the natural antisymmetric ordering yields the quantum Hamiltonian[4]

$$\hat{H}_f = -\alpha^\dagger\alpha - \kappa\,, \tag{2.13}$$

where

$$\alpha = \left(\partial_{\bar{\zeta}} - \kappa\zeta\right)\,, \qquad \alpha^\dagger = \left(\partial_\zeta - \kappa\bar{\zeta}\right)\,. \tag{2.14}$$

These operators satisfy the anticommutation relations

$$\{\alpha, \alpha^\dagger\} = -2\kappa\,. \tag{2.15}$$

The Hamiltonian $\hat{H}_f$ has four linear independent eigenfunctions $\Psi(\zeta, \bar{\zeta})$. Two, which we denote collectively by Ψ_-, have energy $-\kappa$ and the other two, which we denote collectively by Ψ_+, have energy $+\kappa$. From the requirement that Ψ_- is annihilated by α and Ψ_+ is annihilated by $\alpha^\dagger$, it can be seen that

$$\begin{aligned}\Psi_- &= A_-\left(1 + \kappa\bar{\zeta}\zeta\right) + B_-\zeta\,,\\ \Psi_+ &= A_+\left(1 - \kappa\bar{\zeta}\zeta\right) + B_+\bar{\zeta}\,.\end{aligned} \tag{2.16}$$

Note that Ψ_+ can be viewed as an excited state generated by the creation operator $\alpha^\dagger$ from the "vacuum" Ψ_-.

[4]By changing the sign of κ and interchanging the roles of annihilation and creation operators, this could be brought to the form $H = \alpha^\dagger\alpha - \kappa$, which is the standard form for a fermionic oscillator. However, the formulation given here is the one most convenient for the purpose of combining it with the standard Landau model to get the superplane Landau model that we consider here.

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Up to an overall factor, which we may choose at our convenience, the natural inner product on wavefunctions (invariant under translations and phase rotations of ζ) is

$$\langle \Psi_1, \Psi_2 \rangle = \partial_\zeta \partial_{\bar\zeta} \left(\Psi_1^* \Psi_2\right) . \tag{2.17}$$

It is straightforward to verify that wavefunctions with different energies are orthogonal with respect to this inner product, and that

$$\begin{aligned} \langle \Psi_-, \Psi_- \rangle &= 2\kappa \bar{A}_- A_- + \bar{B}_- B_- \,, \\ \langle \Psi_+, \Psi_+ \rangle &= -2\kappa \bar{A}_+ A_+ - \bar{B}_+ B_+ \,. \end{aligned} \tag{2.18}$$

In arriving at this result we have been careful not to assume any particular Grassmann-parity for the complex constants A and B. It would be possible to suppose that all are Grassmann even, in which case it is clear that if the states of Ψ_- have positive norm then the states of Ψ_+ have negative norm. If instead one assumes that Ψ_- and Ψ_+ have a definite Grassmann parity, so that either the A or the B coefficient is Grassmann-odd, then it is still true that the states of Ψ_- have non-negative norm (this now being a complex supernumber) while those of the higher level have a non-positive norm, with some state of negative norm, and this is true whatever assumption one makes about the Grassmann parity of $\Psi_\pm$. Thus, *only Ψ_- has all states of non-negative norm.*

As for the standard Landau model, one can take a limit in which only the lowest Landau level survives. The corresponding LLL Lagrangian is just the fermion WZ term. This is the simplest case of the "odd-coset" models studied in [7], with a Hilbert space spanned by the two positive-norm states.

Before moving on, we pause to comment on the limit in which $\kappa = 0$. The bosonic Landau model becomes a model for a particle moving freely on the complex plane. In contrast, the fermionic Landau model is unphysical when $\kappa = 0$ because the Hamiltonian operator $\hat{H}_f$ is then nilpotent and hence non-diagonalizable. For this reason we henceforth consider only $\kappa \neq 0$. Although this restriction is unphysical in the Landau model, where κ is proportional to the magnetic field, it may be physical in any context in which the fermionic Landau model plays a role since the parameter κ may then have some other interpretation.

2.2 The superplane model and its symmetries

We now return to the Landau model for a particle on the superplane. The Hamiltonian form of the Lagrangian is

$$L_0 = \left(\dot{z} p - i\dot{\zeta}\pi\right) + c.c. - (H_b + H_f)\,. \tag{2.19}$$

The quantum Hamiltonian has energy levels $2\kappa N$ for non-negative integer N. In particular the states $|LLL\rangle$ of the LLL have zero energy and satisfy

$$a|LLL\rangle = 0\,, \qquad \alpha|LLL\rangle = 0\,. \tag{2.20}$$

All these states have positive norm. The first exited states (with energy 2κ) are linear combinations of states of the form $a^\dagger|LLL\rangle$, which all have positive norm, and states of the form $\alpha^\dagger|LLL\rangle$, some of which have negative norm. Thus, only the LLL has all states of positive norm.

Note that the zero point energy cancels between the bosonic and fermionic sectors, as happens in supersymmetric quantum mechanics. However, the "supersymmetry" of the superplane Landau model is rather different from that of supersymmetric quantum mechanics. As for any quadratic Lagrangian (except those with *only* Grassmann-odd variables [7]), the full symmetry group is infinite-dimensional. However, the symmetries of relevance here are those inherited from the supersphere. These are the super-translations of the superplane, the SU(1|1) super-rotations, and an independent U(1) phase rotation.

The super-translation transformations are

$$\begin{pmatrix} \delta z \\ \delta\zeta \end{pmatrix} = \begin{pmatrix} c \\ \gamma \end{pmatrix}, \qquad \begin{pmatrix} \delta p \\ \delta\pi \end{pmatrix} = \kappa \begin{pmatrix} -i\bar{c} \\ \bar{\gamma} \end{pmatrix}, \tag{2.21}$$

for complex constant c and complex Grassmann-odd constant γ. This symmetry is generated by the operators

$$\begin{aligned} P &= -i\left(\partial_z + \kappa\bar{z}\right), \qquad P^\dagger = -i\left(\partial_{\bar{z}} - \kappa z\right) \\ \Pi &= \partial_\zeta + \kappa\bar{\zeta}, \qquad \Pi^\dagger = \partial_{\bar{\zeta}} + \kappa\zeta\,. \end{aligned} \tag{2.22}$$

Their non-zero (anti)commutation relations are

$$[P, P^\dagger] = 2\kappa\,, \qquad \{\Pi^\dagger, \Pi\} = 2\kappa\,. \tag{2.23}$$

Thus, κ is a central charge. We will call the superalgebra defined by these relations the "magnetic translation superalgebra".

The SU(1|1) super-rotation transformations are

$$\begin{pmatrix} \delta z \\ \delta\zeta \end{pmatrix} = \begin{pmatrix} i\theta & -\bar{\epsilon} \\ -\epsilon & i\theta \end{pmatrix}\begin{pmatrix} z \\ \zeta \end{pmatrix}, \qquad \begin{pmatrix} \delta p \\ \delta\pi \end{pmatrix} = \begin{pmatrix} -i\theta & -i\epsilon \\ -i\bar{\epsilon} & -i\theta \end{pmatrix}\begin{pmatrix} p \\ \pi \end{pmatrix} \tag{2.24}$$

for constant angle θ and complex Grassmann-odd parameter ϵ. The odd transformations are generated by the operators

$$Q = z\partial_\zeta - \bar{\zeta}\partial_{\bar{z}}\,, \qquad Q^\dagger = \bar{z}\partial_{\bar{\zeta}} + \zeta\partial_z \tag{2.25}$$

and the even transformation is generated by the Hermitian operator

$$C = z\partial_z + \zeta\partial_\zeta - \bar{z}\partial_{\bar{z}} - \bar{\zeta}\partial_{\bar{\zeta}}\,. \tag{2.26}$$

The only non-zero (anti)commutation relations of these generators is

$$\{Q, Q^\dagger\} = C\,. \tag{2.27}$$

This is analogous to a standard supersymmetry algebra but with C as the Hamiltonian. It should be noted, however, that many of the usual consequences of supersymmetry would not apply anyway because of the negative-norm states.

The SU(1|1) charges, together with the supertranslation charges, span a semi-direct product superalgebra which we will call ISU(1|1). In particular,

$$[Q, P] = i\Pi\,, \qquad \{Q^\dagger, \Pi\} = iP\,, \qquad [C, P] = -P\,, \qquad [C, \Pi] = -\Pi\,. \tag{2.28}$$

However, as shown by (2.23), we must include a central charge $Z = \kappa$; this generates an abelian group, which we call $\mathcal{Z}$ and include as part of the definition of ISU(1|1). The superplane can now be viewed as the coset superspace $\mathrm{ISU}(1|1)/[\mathrm{SU}(1|1) \times \mathcal{Z}]$.

Finally we have an independent U(1) phase rotation with infinitesimal transformations

$$\begin{aligned} \delta z &= i\varphi z\,, \qquad \delta p = -i\varphi p\,, \\ \delta\zeta &= -i\varphi\zeta\,, \qquad \delta\pi = i\varphi\pi\,. \end{aligned} \tag{2.29}$$

This is generated by the Hermitian operator

$$J = \frac{1}{2}\left[z\partial_z - \zeta\partial_\zeta - \bar{z}\partial_{\bar{z}} + \bar{\zeta}\partial_{\bar{\zeta}}\right] \tag{2.30}$$

which has the following non-zero commutation relations with the generators of ISU(1|1)

$$[J, Q] = Q\,, \qquad [J, Q^\dagger] = -Q^\dagger\,, \qquad [J, P] = -P\,, \qquad [J, \Pi] = \Pi\,. \tag{2.31}$$

The supergroup generated by the five even charges $(P, P^\dagger, C, J, Z)$ and the four odd charges $(\Pi, \Pi^\dagger, Q, Q^\dagger)$ will be called IU(1|1), and the superplane can be viewed as the coset superspace $\mathrm{IU}(1|1)/[\mathrm{U}(1|1) \times \mathcal{Z}]$, as mentioned in the introduction. This has the advantage that IU(1|1) is a contraction of SU(2|1), as we now show.

2.3 IU(1|1) as contraction of SU(2|1)

We now sketch how the algebra of the supergroup IU(1|1) defined by the relations (2.23), (2.27), (2.28) and (2.31) can be reproduced as a contraction of the superalgebra $su(2|1)$. The contraction procedure is similar to the one relating $su(2)$ to the algebra of magnetic translations [8].

The bosonic body of the superalgebra $su(2|1)$ is $su(2) \oplus u(1)$ with the generators $J_\pm, J_3$ and B [5]

$$\begin{aligned} &[J_+, J_-] = -J_3\,,\ [J_3, J_\pm] = \pm 2J_\pm\,, \quad [B, J_3] = 0\,,\ [B, J_\pm] = \mp J_\pm\,, \\ &J_3^\dagger = J_3\,,\ B^\dagger = B\,,\ J_+^\dagger = -J_-\,. \end{aligned} \tag{2.32}$$

The odd sector is spanned by an SU(2) doublet generators $S_1, S_2, \bar{S}^1, \bar{S}^2$ with the following non-vanishing (anti)commutation relations (and their conjugates):

$$\begin{aligned} &\{S_1, \bar{S}^1\} = J_3 + B\,,\ \{S_2, \bar{S}^2\} = B\,,\ \{S_1, \bar{S}^2\} = -J_+\,,\ \{S_2, \bar{S}^1\} = J_-\,, \\ &[J_3, S_1] = S_1\,,\ [J_3, S_2] = -S_2\,,\ [B, S_1] = -Q_1\,,\ [B, S_2] = 0\,, \\ &[J_+, S_1] = 0\,,\ [J_+, S_2] = -S_1\,,\ [J_-, S_1] = S_2\,,\ [J_-, S_2] = 0\,. \end{aligned} \tag{2.33}$$

Note that the second U(1) generator B basically has the same commutation relations with $J_\pm$ as J_3,[5] but both these generators (B and J_3) have different action on the spinors.

[5] This set of generators can be split into the mutually commuting $u(1)$ and $su(2)$ sets by passing to the appropriate linear combination of B and J_3, but we prefer to use this basis in order to have a correspondence with the notation of ref. [5].

The contraction leading to the magnetic translation superalgebra introduced in the previous subsection goes as follows. Firstly one redefines (and/or rename) the generators as

$$J_3 = 2n - 2J\,, \quad J_+ = iR\,P\,, \quad J_- = iR\,P^\dagger\,, \quad S_1 = R\,\Pi\,, \quad S_2 = -Q\,, \quad B = C\,, \tag{2.34}$$

where n and R are two real parameters (R is a radius of the sphere $S^2 \sim \mathrm{SU}(2)/\,\mathrm{U}(1)$ while n, in the dynamical framework of a particle moving on the superflag manifold $\mathrm{SU}(2|1)/[\mathrm{U}(1) \times \mathrm{U}(1)]$ [5], acquires a nice meaning of the strength of the $\mathrm{SU}(2)/\,\mathrm{U}(1)$ WZW term). Then one substitutes this into (2.33) and let $R \to \infty$, assuming that

$$\frac{n}{R^2} \equiv \kappa < \infty\,. \tag{2.35}$$

As the result of this contraction procedure, the algebra of the $su(2)$ generators $J_\pm, J_3$ in (2.33) goes over into the magnetic translation algebra (given by the first relation in (2.23)) and the relations (2.33) become just (2.27), (2.28) and the second relation in (2.23) (plus the evident additional commutation relations with the generator J, eq. (2.31)). It is worth noting that, in the contraction limit, one of the U(1) charges, J, fully decouples and generates an outer U(1) automorphism, while $B \equiv C$ still remains in the r.h.s. of $\{S_2, \bar S^2\}$. Another notable feature is the appearance of the constant central charge κ which thus formally extends the full number of bosonic generators to five as compared with four such generators in SU(2|1); this also happens in the purely bosonic $su(2)$ or $sl(2,\mathbb{R})$ cases [8].

3. The planar superflag Landau model

The problem with the Landau model on the superplane is that the second-order Lagrangian for the Grassmann-odd variable implies the presence of ghosts (negative norm states) in the quantum theory. This is forced by the Q-supersymmetry of SU(1|1) that relates bosons to fermions, so any solution to this problem would appear to require a breaking of this symmetry, but we would need the breaking to be spontaneous in order to maintain the IU(1|1) symmetry of the Lagrangian. This suggests that we aim for a non-linear realization of the Q-supersymmetry by introducing a Goldstino variable ξ with the Q-transformation

$$\delta\xi = \epsilon\,. \tag{3.1}$$

We now observe that the new Lagrangian

$$\tilde L = L_0 - \left(|\dot z|^2 + \dot\zeta\dot{\bar\zeta}\right)\left(\xi + \dot\zeta/\dot z\right)\left(\bar\xi + \dot{\bar\zeta}/\dot{\bar z}\right) \tag{3.2}$$

is invariant under all the symmetries previously established for L_0. Collecting terms, we have

$$\tilde L = \left(1 + \bar\xi\xi\right)|\dot z|^2 + \left(\bar\xi\dot{\bar z}\dot\zeta - \xi\dot z\dot{\bar\zeta}\right) + \bar\xi\xi\dot\zeta\dot{\bar\zeta} - i\kappa\left(\dot z\bar z - \dot{\bar z}z + \dot\zeta\bar\zeta + \dot{\bar\zeta}\zeta\right)\,, \tag{3.3}$$

which shows both that the new Lagrangian is well-defined at $\dot z = 0$, despite initial appearances, *and* that the second-order kinetic term $\dot\zeta\dot{\bar\zeta}$ term now has a nilpotent coefficient. The implications of this are not immediately apparent but will become clear in due course.

Although it might appear that we have now solved, or at least ameliorated, the problem of ghosts, we have actually just hidden it; the ξ equation of motion is

$$\left(\dot{z}\dot{\bar{z}} + \dot{\zeta}\dot{\bar{\zeta}}\right)\xi + \dot{\bar{z}}\dot{\zeta} = 0 \tag{3.4}$$

and if $\dot{z} \neq 0$ this implies

$$\xi = -\frac{\dot{\zeta}}{\dot{z}}\,. \tag{3.5}$$

Back substitution into $\tilde{L}$ yields the quadratic Lagrangian L_0 with which we started, so $\tilde{L}$ is classically equivalent to L_0, except possibly when $\dot{z} = 0$, which implies zero classical energy. Thus, apart from this subtlety, to which we return later, nothing has yet been accomplished. However, there is now an additional WZ term that we can add to the Lagrangian arising from the closed invariant 2-form $d\bar{\xi} \wedge d\xi$. This leads us to the Lagrangian

$$\begin{aligned} L = &\left(1 + \bar{\xi}\xi\right)|\dot{z}|^2 + \left(\bar{\xi}\dot{\bar{z}}\dot{\zeta} - \xi\dot{z}\dot{\bar{\zeta}}\right) + \bar{\xi}\xi\dot{\zeta}\dot{\bar{\zeta}} \\ &- i\kappa\left(\dot{z}\bar{z} - \dot{\bar{z}}z + \dot{\zeta}\bar{\zeta} + \dot{\bar{\zeta}}\zeta\right) + iM\left(\bar{\xi}\dot{\xi} + \xi\dot{\bar{\xi}}\right) \end{aligned} \tag{3.6}$$

for some constant M. This model is actually the planar limit of the superflag Landau model of [5].

We now proceed to a detailed analysis of this model, in its Hamiltonian formulation, first classically and then quantum-mechanically. We then provide a more geometrical derivation of our results based on the theory of non-linear realizations.

3.1 Hamiltonian analysis

Introducing the complex Grassmann-odd momentum χ conjugate to ξ, the Hamiltonian form of the Lagrangian (3.6) is[6]

$$L = \left[\dot{z}\tilde{p} - i\dot{\zeta}\pi - i\dot{\xi}\chi + \lambda_\zeta\varphi_\zeta + \lambda_\xi\varphi_\xi\right] + c.c. - H\,, \tag{3.7}$$

where the Hamiltonian is

$$H = \left(1 - \bar{\xi}\xi\right)|\tilde{p} + i\kappa\bar{z}|^2 \tag{3.8}$$

and the complex Grassmann-odd variables λ_ζ and λ_ξ are Lagrange multipliers for the "fermionic" constraints $\varphi_\zeta \approx 0$ and $\varphi_\xi \approx 0$ (in Dirac's "weak equality" notation). The constraint functions are

$$\varphi_\zeta = \pi - \kappa\bar{\zeta} + i\bar{\xi}\left(\tilde{p} + i\kappa\bar{z}\right), \qquad \varphi_\xi = \chi - M\bar{\xi}\,. \tag{3.9}$$

To establish the equivalence of (3.7) to (3.6) we solve the constraints to reduce (3.7) to

$$\begin{aligned} L = &\left\{\left[\dot{z}\tilde{p} - i\kappa\,\dot{\zeta}\bar{\zeta} - iM\dot{\xi}\bar{\xi}\right] + c.c\right\} - |\tilde{p} + i\kappa\bar{z}|^2 - \dot{\bar{\zeta}}\dot{\zeta} \\ &+ \left[\left(\tilde{p} + i\kappa\bar{z}\right)\bar{\xi} + \dot{\bar{\zeta}}\right]\left[\left(\bar{\tilde{p}} - i\kappa z\right)\xi + \dot{\zeta}\right]. \end{aligned} \tag{3.10}$$

Elimination of $\tilde{p}$ now yields (3.6).

[6] We here denote by $\tilde{p}$ the momentum conjugate to z to distinguish it from the momentum conjugate to z in a different set of variables that we will later use to quantize the model.

The occurrence of fermion constraints is to be expected in any model with canonical, first-order, fermion kinetic terms, and these constraints are normally second class, in Dirac's terminology. Here, however, we have an additional "bodyless" second-order fermion kinetic term, and this has a curious consequence. A computation shows that although the Poisson bracket of the analytic constraint functions $(\varphi_\zeta, \varphi_\xi)$ is zero, the matrix of Poisson brackets of these functions with their complex conjugates is non-zero. In fact,

$$\det \begin{pmatrix} \{\varphi_\zeta, \bar{\varphi}_\zeta\}_{PB} & \{\varphi_\zeta, \bar{\varphi}_\xi\}_{PB} \\ \{\varphi_\xi, \bar{\varphi}_\zeta\}_{PB} & \{\varphi_\xi, \bar{\varphi}_\xi\}_{PB} \end{pmatrix} = \left(1 + \bar{\xi}\xi\right) [H - 4\kappa M] \,. \tag{3.11}$$

It follows that the constraints *considered together with their complex conjugates* are second class everywhere *except on the surface* $H = 4\kappa M$; on this surface there are first class constraints.

This unusual state of affairs merits a more detailed analysis. We begin with the $M = 0$ case, for which the energy surface $H = 4\kappa M$ reduces to the point $H = 0$. As long as the classical energy $(1-\bar{\xi}\xi)\, |\tilde{p} + i\kappa\bar{z}|^2$ (and hence $|\tilde{p} + i\kappa\bar{z}|^2$) is non-zero we may treat ξ in (3.10) as an auxiliary variable that can be eliminated by its equation of motion

$$(\tilde{p} + i\kappa\bar{z}) \left[(\bar{\tilde{p}} - i\kappa z)\xi + \dot{\zeta}\right] = 0 \,. \tag{3.12}$$

This is equivalent to

$$\xi = -\dot{\zeta} / \left(\bar{\tilde{p}} - i\kappa z\right) \tag{3.13}$$

provided that $|\tilde{p} + i\kappa\bar{z}|^2 \neq 0$. After substitution for ξ in (3.10), and subsequent elimination of the momentum variable $\tilde{p}$, we recover the Lagrangian of the superplane Landau model. This confirms our analysis of the previous subsection, but now it is clear how to proceed when the classical energy vanishes; in this case $\tilde{p} = -i\kappa\bar{z}$ and the Lagrangian (3.10) becomes[7]

$$L_0 = -i\kappa \left\{ \dot{z}\bar{z} - z\dot{\bar{z}} + \dot{\zeta}\bar{\zeta} + \dot{\bar{\zeta}}\zeta \right\} . \tag{3.14}$$

This is the LLL Lagrangian for a particle on the superplane; the proof of the equivalence of the superplane model to the $M = 0$ planar superflag model is thus completed.

Let us now consider the case of arbitrary M. The properties of our model on the exceptional energy surface $H = 4\kappa M$ can be studied via a new Lagrangian obtained by imposing $H = 4\kappa M$ as a new, bosonic, constraint via a new Lagrange multiplier variable $e(t)$. The resulting Lagrangian is equivalent to

$$\begin{aligned} L = &\left[\dot{z}\tilde{p} - i\dot{\zeta}\pi - i\dot{\xi}\chi + \lambda_\zeta\varphi_\zeta + \lambda_\xi\varphi_\xi\right] + c.c. - 4\kappa M \\ &- e\left[|\tilde{p} + i\kappa\bar{z}|^2 - 4\left(1 + \bar{\xi}\xi\right)\kappa M\right] . \end{aligned} \tag{3.15}$$

This action is time-reparametrization invariant, with e as the einbein. Moreover, as should be clear from its construction, this action also has a hidden fermionic gauge invariance. In this respect, it is analogous to the superparticle action with its hidden "kappa-symmetry", the constraint $H = 4\kappa M$ being analogous to the standard mass-shell superparticle condition

[7]Note that the variables (z, ζ) are still independent and off-shell.

with $2\sqrt{\kappa M}$ as a "mass". Many methods have been developed to deal with the mixed first and second class fermionic constraints of the superparticle, and these could be applied here. Perhaps the simplest is just to solve all the constraints to obtain a physical phase-space Lagrangian, and that is what we will do here.

The fermionic constraints are trivially solved for the fermionic momenta (π, χ). The new bosonic constraint $H = 4\kappa M$ has the general solution

$$\tilde{p} + i\kappa\bar{z} = 2e^{i\phi}\left(1 + \frac{1}{2}\bar{\xi}\xi\right)\sqrt{\kappa M}\,, \tag{3.16}$$

for some arbitrary phase $\phi(t)$. Using this to eliminate $\tilde{p}$ in favour of ϕ, we arrive at the Lagrangian

$$\begin{aligned} L_{4\kappa M} = &-i\kappa\left(\dot{z}\bar{z} - z\dot{\bar{z}} + \dot{\zeta}\bar{\zeta} - \zeta\dot{\bar{\zeta}}\right) + 2\left(1 + \frac{1}{2}\bar{\xi}\xi\right)\sqrt{\kappa M}\left[e^{i\phi}\left(\dot{z} + \bar{\xi}\dot{\zeta}\right) + c.c.\right] \\ &+ iM\left(\bar{\xi}\dot{\xi} + \xi\dot{\bar{\xi}}\right) - 4\kappa M. \end{aligned} \tag{3.17}$$

The new phase variable ϕ is actually a gauge variable for the U(1) gauge invariance with infinitesimal gauge transformations

$$\delta\phi = a(t)\,, \qquad \delta z = \sqrt{\frac{M}{\kappa}}\left(1 + \frac{1}{2}\bar{\xi}\xi\right)e^{-i\phi}\,a(t)\,, \qquad \delta\zeta = -\sqrt{\frac{M}{\kappa}}\,\xi e^{-i\phi}\,a(t)\,, \tag{3.18}$$

where $a(t)$ is the U(1) gauge parameter. This gauge invariance allows us to set $\phi(t) = 0$. Much more remarkable is the *fermionic* gauge invariance with infinitesimal gauge transformations

$$\delta\xi = \omega\,, \quad \delta\zeta = -i\sqrt{\frac{M}{\kappa}}e^{-i\phi}\omega\,, \qquad \delta z = \frac{i}{2}\sqrt{\frac{M}{\kappa}}e^{-i\phi}\left(\bar{\omega}\xi + \bar{\xi}\omega\right)\,, \tag{3.19}$$

where $\omega(t)$ is the complex anticommuting gauge parameter. This gauge invariance allows us to set $\xi(t) = 0$.

For the gauge choices $\phi = 0$ and $\xi = 0$, the Lagrangian (3.17) reduces to

$$L_{4\kappa M} = -i\kappa\left(\dot{y}\bar{y} - y\dot{\bar{y}} + \dot{\zeta}\bar{\zeta} - \zeta\dot{\bar{\zeta}}\right) - 4\kappa M\,, \tag{3.20}$$

where

$$y = z - i\sqrt{M/\kappa}\,. \tag{3.21}$$

This is again the LLL Lagrangian for the superplane model, as in (3.14), but with the vacuum energy shifted by $4\kappa M$. We shall see later that this result has interesting consequences for the quantum theory when M is an integer.

Before turning to the quantum theory we must address a further technical problem; the Poisson bracket of the Hamiltonian (3.8) with the constraint function φ_ζ is not even weakly zero. This problem could be circumvented by considering[8]

$$H' = \left(1 + \bar{\xi}\xi\right)\left|\tilde{p} + i\kappa\bar{z} + i\xi\left(\pi - \kappa\bar{\zeta}\right)\right|^2, \tag{3.22}$$

[8]Note the change of sign in the prefactor.

which has weakly vanishing Poisson brackets with the constraints and is weakly equal to H. However, this has the disadvantage that H' depends on the fermionic momenta. We prefer to proceed differently. We define the new anticommuting variables

$$\xi^1 = \zeta + z\xi \,, \qquad \xi^2 = \xi \,, \tag{3.23}$$

and let (χ_1, χ_2) be their canonically conjugate momenta. Defining

$$p = \tilde{p} + i\xi\pi \,, \tag{3.24}$$

we find that the Lagrangian in the new variables is

$$L = \left[\dot{z}p - i\dot{\xi}^i\chi_i + \lambda^i\varphi_i\right] + c.c. - H \,, \tag{3.25}$$

where λ^i are Lagrange multipliers for the constraints $\varphi_i \approx 0$ $(i = 1, 2)$. The constraint functions are

$$\begin{aligned} \varphi_1 &= \chi_1 - \kappa\bar{\xi}_1\left(1 - \bar{\xi}_2\xi^2\right) + i\bar{\xi}_2 p \,, \\ \varphi_2 &= \chi_2 + \kappa z\bar{\xi}_1\left(1 - \bar{\xi}_2\xi^2\right) - i\bar{\xi}_2 z p - M\bar{\xi}_2 \,, \end{aligned} \tag{3.26}$$

and the Hamiltonian is now

$$H = \left(1 + \bar{\xi}^2\xi_2\right)\left|p + i\kappa\bar{z} - i\kappa\xi^2\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\right|^2 . \tag{3.27}$$

This Hamiltonian has (strongly) vanishing Poisson brackets with the constraints. As before, all these constraints are second class except on the surface $H = 4\kappa M$.

3.2 Quantization

We will quantize the planar superflag model of the previous section using the Gupta-Bleuler method; details and references can be found in our previous papers [7, 4, 5]. This is a method of quantization in the presence of analytic constraints that are second class only when considered in conjunction with their complex conjugates, exactly as we found for the constraints of the planar superflag model. We also found that there is a surface on which these constraints are not second class, but we will deal with this problem when and where it presents a difficulty. We also work with the variables (z, ξ^1, ξ^2) in this section.

The method instructs us to quantize initially as there were no constraint, so we make the usual replacements

$$p \to \hat{p} = -i\partial_z \,, \qquad \bar{p} \to -i\partial_{\bar{z}} \,, \qquad \chi_i \to \hat{\chi}_i = \partial_{\xi^i} \,, \qquad \bar{\chi}^i = \partial_{\bar{\xi}_i} \,. \tag{3.28}$$

The Hamiltonian can be written in terms of the operators

$$\nabla_z = \partial_z - \kappa\bar{z} + \kappa\xi^2\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right) , \qquad \nabla_{\bar{z}} = \partial_{\bar{z}} + \kappa z + \kappa\bar{\xi}_2\left(\xi^1 - z\xi^2\right) , \tag{3.29}$$

which satisfy

$$[\nabla_z, \nabla_{\bar{z}}] = 2\kappa\left(1 - \bar{\xi}_2\xi^2\right) . \tag{3.30}$$

There is an operator ordering ambiguity in the quantum Hamiltonian, but this affects only the choice of ground state energy. If we resolve this ambiguity in the usual way we arrive at the Hamiltonian operator

$$\hat{H} = -\frac{1}{2}\left(1+\bar{\xi}_2\xi^2\right)\{\nabla_z, \nabla_{\bar{z}}\} = -\left(1+\bar{\xi}_2\xi^2\right)\nabla_z\nabla_{\bar{z}} + \kappa\,. \tag{3.31}$$

This operator $\hat{H}$ is positive definite. As we shall shortly see, the lowest eigenvalue of $\hat{H}$ is κ, so the cancellation of vacuum energies that we noted for the superplane model no longer occurs. This is because the Hamiltonian no longer depends on ζ. This raises a puzzle because the vacuum energy of the $M = 0$ planar superflag model is also equal to κ, but this model is classically equivalent to the superplane model. There is thus an apparent quantum inequivalence of the $M = 0$ planar superflag model with the superplane Landau model, but this is a trivial difference that could be removed by a different operator ordering prescription. As we shall see, the equivalence holds quantum mechanically in all other respects.

The constraints are now taken into account by the physical state conditions

$$\hat{\bar{\varphi}}^i\Psi = 0 \qquad (i = 1, 2)\,, \tag{3.32}$$

where

$$\begin{aligned}
\hat{\bar{\varphi}}^1 &= \partial_{\bar{\xi}_1} - \kappa\xi^1\left(1-\bar{\xi}_2\xi^2\right) - \xi^2\partial_{\bar{z}}\,,\\
\hat{\bar{\varphi}}^2 &= \partial_{\bar{\xi}_2} + \kappa\bar{z}\xi^1\left(1-\bar{\xi}_2\xi^2\right) + \xi^2\bar{z}\partial_{\bar{z}} - M\xi^2\,.
\end{aligned} \tag{3.33}$$

Solving these constraints one finds that physical wavefunctions have the form

$$\Psi = K\,\Phi\left(z, \bar{z}_{sh}, \xi^1, \xi^2\right)\,, \qquad \bar{z}_{sh} = \bar{z} - \xi^2\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\,, \tag{3.34}$$

where K is a real prefactor which we write as

$$K = K_1^M\, e^{-\kappa K_2} \tag{3.35}$$

with

$$K_1 = \left(1+\bar{\xi}_2\xi^2\right)\,, \qquad K_2 = \left[|z|^2 + \left(\xi^1 - z\xi^2\right)\left(\bar{\xi}_1 - \bar{z}\bar{\xi}_2\right)\right]\,. \tag{3.36}$$

Thus, physical states are described by "chiral " wavefunctions $\Phi\left(z, \bar{z}_{sh}, \xi^1, \xi^2\right)$ (we use this term because of the close analogy to chiral superfields in supersymmetric field theories). Observe that

$$\nabla_{\bar{z}}\Psi = K\partial_{\bar{z}}\Phi\,, \qquad \nabla_z\Phi = K\,\tilde{\nabla}_z\Phi\,, \tag{3.37}$$

where

$$\tilde{\nabla}_z = \partial_z - 2\kappa\bar{z}_{sh}\,. \tag{3.38}$$

This derivative has the property that it preserves chirality by taking a chiral wavefunction to another chiral wavefunction. It follows that the differential operators $(\nabla_z, \nabla_{\bar{z}})$ become the differential operators $(\tilde{\nabla}_z, \partial_{\bar{z}})$ in the chiral basis, i.e., when acting on reduced wavefunctions. In particular the hamiltonian operator $\hat{H}$ is replaced by

$$\hat{H}_{red} = -K_1\tilde{\nabla}_z\partial_{\bar{z}} + \kappa \tag{3.39}$$

in the chiral basis.

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Reduced ground state wavefunctions, of energy *kappa*, are *analytic*, so ground state wavefunctions have the form

$$\Psi^{(0)} = K\, \Phi_0^{(0)}(z, \xi^1, \xi^2)\,. \tag{3.40}$$

One can now generate an infinite set of eigenvectors of $\hat{H}$ by considering:

$$\Psi^{(N)} = \nabla_z^N \left[K \Phi_0^{(N)}\left(z, \xi^i\right)\right] = K \tilde{\nabla}_z^N \Phi_0^{(N)}\left(z, \xi^i\right). \tag{3.41}$$

Indeed, using the commutation relation

$$\left[\partial_{\bar{z}}, \tilde{\nabla}_z^N\right] = -2\kappa N K_1^{-1} \tilde{\nabla}_z^{N-1}\,, \tag{3.42}$$

it can be seen that

$$\hat{H}_{red}\left(\tilde{\nabla}_z^N \Phi_0^{(N)}\right) = 2\kappa \left(N + \frac{1}{2}\right)\left(\tilde{\nabla}_z^N \Phi_0^{(N)}\right), \tag{3.43}$$

and hence that the wavefunctions (3.41) are eigenfunctions of $\hat{H}$ with energy $2\kappa\left(N+\frac{1}{2}\right)$. Note that $\tilde{\nabla}_z$ preserves chirality, but not the analyticity, so the reduced function $\Phi^{(N)} = \tilde{\nabla}_z^N \Phi_0^{(N)}(z, \xi^1, \xi^2)$ is a particular case of Φ defined in (3.34), with a special dependence on $\bar{z}_{sh}$. Note also that the analytic "ground state" functions $\Phi_0^{(N)}$ for different N differ in their "external" C charge $\tilde{C} = 2M - N$. The wavefunctions $\Psi^{(N)}$ and $\Phi^{(N)}$ have the fixed charge $\tilde{C} = 2M$ for any N, since ∇_z and $\tilde{\nabla}_z$ carry $\tilde{C} = 1$ (see subsection 3.3).

We have now found the energy eigenstates so we turn to the question of their norm. The integration measure

$$d\mu = dz d\bar{z} \partial_{\bar{\xi}_1} \partial_{\xi^1} \partial_{\bar{\xi}_2} \partial_{\xi^2} \tag{3.44}$$

is invariant under the symmetries of the model established previously, so we define the norm of Ψ by

$$|||\Psi|||^2 = \int d\mu\, |\Psi|^2 = \int d\mu\, K_1^{2M} e^{-2\kappa K_2}\, |\Phi|^2\,. \tag{3.45}$$

For a ground state, the reduced wavefunction is analytic and can be expanded as

$$\Phi_0^{(0)} = A^{(0)} + \xi^i \psi_i^{(0)} + F^{(0)} \xi^1 \xi^2\,, \tag{3.46}$$

where all the coefficients are functions of z. A calculation shows that its norm is

$$|||\Phi_0^{(0)}|||^2 = 4\kappa M ||A^{(0)}||^2 + 2M||\psi_1^{(0)}||^2 + 2\kappa||\psi_2^{(0)} + z\psi_1^{(0)}||^2 + ||F^{(0)}||^2\,, \tag{3.47}$$

where

$$||f||^2 = \int dz d\bar{z}\, e^{-2\kappa |z|^2}\, |f(z,\bar{z})|^2 \tag{3.48}$$

for any function f on the complex plane. Note that we have a shortened multiplet when $M = 0$ because there are then states with zero norm. This is the quantum manifestation of the classical observation that for $M = 0$ the constraints are not all second class when $H = 0$.

Consider now the first excited states, at $N = 1$. Integrating by parts with respect to $\partial_z, \partial_{\bar{z}}$, one sees that

$$|||\Psi^{(1)}|||^2 = 2\kappa \int d\mu\, K_1^{2M-1} e^{-2\kappa K_2} |\Phi_0^{(1)}|^2 \,. \tag{3.49}$$

In other words, the coefficient M is shifted downwards by 1/2. Similarly,

$$|||\Psi^{(N)}|||^2 = (2\kappa)^N N! \int d\mu\, K_1^{2M-N} e^{-2\kappa K_2} |\Phi_0^{(N)}|^2 \,, \tag{3.50}$$

so the coefficient M is shifted downwards by $N/2$ at level N. It follows that $|||\Psi^{(N)}|||$ is also given by the formula (3.47), apart from the numerical factor $(2\kappa)^N N!$, but with $2M \to 2M - N$. Thus, negative contributions to the norm must appear for $N > 2M$. If $2M$ is a positive integer then the highest level without negative norm states is the $(2M+1)$th level with $N = 2M$, but this level has zero norm states, as for $M = 0$. The states at this level will therefore form short supermultiplets as only the components $\psi_2^{(2M)} + z\psi_1^{(2M)}$, $F^{(2M)}$ contribute to $|||\Psi^{(N=2M)}|||$. The energy of the $N = 2M$ level for integer $2M$ is $4\kappa M + \kappa$. Apart from the quantum shift by κ noted earlier, this is just the energy of the exceptional energy surface $H = 4\kappa M$ of the classical theory. Zero norm states in the quantum theory at this level are what one expects from the fermionic gauge invariance at this level.

Just as one can discard all excited states of the supersphere, or superplane, Landau model to arrive at a perfectly physical LLL model, so we can discard all states in the $N > 2M$ Landau levels of the superflag, or planar superflag, models to arrive at a physical model described by the LLL together with the first N excited levels. This remains true when $2M$ is not an integer (provided it is positive), the only difference being that the top level, with $N = [2M]$, has no zero norm states.

3.3 Geometrical interpretation

So far we have used a direct algebraic analysis because our aim has been to show how the results of our previous paper on the superflag Landau model can be understood very explicitly in the planar limit, without any elaborate formalism. However, we now develop a geometrical interpretation in terms of superfields on the coset superspace

$$\mathcal{K} = \mathrm{IU}(1|1)/[\mathrm{U}(1) \times \mathrm{U}(1) \times \mathcal{Z}] \,. \tag{3.51}$$

Recall that $\mathcal{Z}$ is the group generated by the "magnetic" central charge Z, which we identify with the constant κ.

The coset representative in the appropriate exponential parametrization can be written in terms of coordinates (u, η^1, η^2) as

$$g = e^{\mathcal{A}_1} e^{\mathcal{A}_2} \,, \tag{3.52}$$

where[9]

$$\mathcal{A}_1 = \eta^1 \Pi - \eta^2 Q + \bar{\eta}_1 \Pi^\dagger - \bar{\eta}_2 Q^\dagger \,, \quad \mathcal{A}_2 = -iuP - i\bar{u}P^\dagger \,, \tag{3.53}$$

[9]We take the Grassmann-odd coordinates η^i to anticommute with the odd charges. One can equally well take them to commute with the odd charges because with an appropriate change in the definition (3.52) one obtains identical results.

where the signs are chosen for later convenience. The coordinates appearing in the above parametrization of the coset superspace are related to the coordinates (z, ζ, ξ) used previously by

$$u = z - \frac{1}{2}\zeta\bar{\xi}\,, \qquad \eta^1 = \zeta + z\xi - \frac{1}{3}\bar{\xi}\xi\,, \qquad \eta^2 = \xi\,. \tag{3.54}$$

The left-covariant Cartan forms and the superconnections on the stability subgroup generated by C and the central charge κ are defined by[10]

$$g^{-1}dg = i\omega_P P + i\bar{\omega}_P P^\dagger + \omega^1\Pi + \bar{\omega}_1\Pi^\dagger - \omega^2 Q - \bar{\omega}_2 Q^\dagger + A_C C + A_{2\kappa}\kappa\,. \tag{3.55}$$

A calculation yields[11]

$$\begin{aligned} &\omega_P = -\left(1 + \frac{1}{2}\bar{\xi}\xi\right) dz - \bar{\xi}d\zeta\,, \qquad \omega^1 = \xi dz + \left(1 - \frac{1}{2}\bar{\xi}\xi\right) d\zeta\,, \qquad \omega^2 = d\xi\,, \\ &A_{2\kappa} = -\left(\bar{z}\,dz - z\,d\bar{z} - \bar{\zeta}\,d\zeta - \zeta\,d\bar{\zeta}\right), \quad A_C = \frac{1}{2}\left(\xi d\bar{\xi} + \bar{\xi}d\xi\right). \end{aligned} \tag{3.56}$$

It is now easy to rewrite the invariant Lagrangians (2.1), (3.2) and (3.6) of the previous sections in a manifestly invariant form in terms of pullbacks of the above Cartan forms:

$$L_0 = |\hat{\omega}_P|^2 + \hat{\omega}^1\hat{\bar{\omega}}_1 + i\kappa\hat{A}_{2\kappa}\,, \quad \tilde{L} = |\hat{\omega}_P|^2 + i\kappa\hat{A}_{2\kappa}\,, \quad L = |\hat{\omega}_P|^2 + i\kappa\hat{A}_{2\kappa} + 2iM\hat{A}_C\,. \tag{3.57}$$

Here the "hat" denotes a pullback. Note that the passage from the superplane Landau model, with Lagrangian L_0, to the $M = 0$ planar superflag model, with Lagrangian $\tilde{L}$, involves the subtraction of the term $\hat{\omega}^1\hat{\bar{\omega}}_1$. The Lagrangian L_0 is necessarily independent of the $\xi, \bar{\xi}$ variables because it is invariant under local SU(1|1) transformations that rotate the forms ω_P and ω^1 (and their conjugates) into each other.

Note also that the equation of motion (3.4) derived from $\tilde{L}$ has the following nice representation in terms of the Cartan forms:

$$\hat{\omega}^1\hat{\bar{\omega}}_P = 0\,. \tag{3.58}$$

This equation has two solutions. One is

$$\hat{\omega}^1 = 0\,, \tag{3.59}$$

which a covariant inverse Higgs-type constraint [9] that is equivalent to (3.5). The other is

$$\hat{\omega}_P = 0 \quad \Rightarrow \quad \dot{z} = -\bar{\xi}\dot{\zeta}\,, \tag{3.60}$$

in which case all other equations of motion are identically satisfied. As we have seen, this second solution reduces the model to its LLL sector.

Finally, we explain the geometric meaning of the wavefunctions $\Psi^{(N)}$ which are eigenvectors of the Hamiltonian $\hat{H}$ defined in (3.31). As a first step, we note that the full

[10] As the second U(1) in the denominator of (3.51) corresponds to an outer automorphism of ISU(1|1) (see (2.30), (2.31)), there appears no connection associated with its generator J.

[11] The $A_{2\kappa}$ connection given here is equivalent to the connection defined by (3.55) but differs from it by a field-dependent gauge transformation.

generators $\hat{Q}$, $\hat{Q}^\dagger$ calculated by the Noether procedure from the Lagrangian L defined in (3.6) are given by

$$\hat{Q} = Q - \frac{\partial}{\partial \xi} - M\bar{\xi}\,, \quad \hat{Q}^\dagger = Q^\dagger - \frac{\partial}{\partial \bar{\xi}} - M\xi\,, \tag{3.61}$$

where $Q, Q^\dagger$ were defined in (2.25). Correspondingly, the full C charge appearing in $\{\hat{Q}, \hat{Q}^\dagger\} = \hat{C}$ is given by

$$\hat{C} = C + 2M \equiv C + \tilde{C}\,, \tag{3.62}$$

where C, the purely differential part of $\hat{C}$, was defined in (2.26). The additional term $\tilde{C} = 2M$ can be interpreted as the "external" C charge of the general wavefunction $\Psi(z,\bar{z},\xi,\bar{\xi},\zeta,\bar{\zeta})$, in accordance with the fact that this function is given on the coset manifold $\mathrm{IU}(1|1)/[\mathrm{U}(1)\times\mathrm{U}(1)\times\mathcal{Z}]$ and can possess non-zero quantum numbers of the stability subgroup. The generator Z acts on Ψ just as the multiplication of the latter by the central charge κ.[12] Thus the wavefunction Ψ carries the "magnetic" central charge κ and the external C charge $\tilde{C} = 2M$.

For the next step we find it convenient to use the parametrization $(z, \bar{z}, \xi^i, \bar{\xi}_i)$ of subsection 3.2. In accord with the standard rules of the nonlinear realizations theory, the covariant differential $\mathcal{D}\Psi$ of Ψ, as well as covariant derivatives of Ψ are defined by the relation

$$\mathcal{D}\Psi = \left(d + A_{2\kappa}\,\kappa + A_C\,\tilde{C}\right)\Psi \equiv -\omega_P \mathcal{D}_z\Psi - \bar{\omega}_P\mathcal{D}_{\bar{z}}\Psi + \omega^i\mathcal{D}_i\Psi + \bar{\omega}_i\bar{\mathcal{D}}^i\Psi\,, \tag{3.63}$$

where the signs were again chosen for further convenience. It is easy to find the explicit form of these covariant derivatives. In particular,

$$\mathcal{D}_z = K_1^{\frac{1}{2}}\nabla_z\,, \quad \mathcal{D}_{\bar{z}} = K_1^{\frac{1}{2}}\nabla_{\bar{z}}\,, \quad \{\mathcal{D}_z, \mathcal{D}_{\bar{z}}\} = 2\kappa\,, \tag{3.64}$$

where $\nabla_z, \nabla_{\bar{z}}$ were defined in (3.29). The covariant spinor derivatives $\bar{\mathcal{D}}^i$ are:

$$\bar{\mathcal{D}}^1 = K_1^{\frac{1}{2}}\left(\frac{\partial}{\partial\bar{\xi}_1} - \xi^2\partial_{\bar{z}} - \kappa\xi^1 K_1^{-1}\right), \quad \bar{\mathcal{D}}^2 = \frac{\partial}{\partial\bar{\xi}_2} + \bar{z}\frac{\partial}{\partial\bar{\xi}_1} - \frac{1}{2}\xi^2\,\tilde{C}. \tag{3.65}$$

They satisfy the following non-zero covariant (anti)commutation relations

$$[\bar{\mathcal{D}}^1, \mathcal{D}_z] = [\bar{\mathcal{D}}^1, \mathcal{D}_{\bar{z}}] = 0\,, \quad [\bar{\mathcal{D}}^2, \mathcal{D}_z] = 0\,, \quad [\bar{\mathcal{D}}^2, \mathcal{D}_{\bar{z}}] = -\bar{\mathcal{D}}^1\,, \tag{3.66}$$

$$\{\bar{\mathcal{D}}^1, \bar{\mathcal{D}}^2\} = 0\,. \tag{3.67}$$

One should take into account that all coset coordinates and their covariant derivatives are inert under the action of the "magnetic" central charge Z which has the non-zero eigenvalue κ only on the wave function Ψ; at the same time, the U(1) charge C has a non-trivial left action on the coset coordinates $z, \bar{z}, \xi^1, \bar{\xi}_1$ as follows from the commutation relations (2.28). Under the above normalization, such that Ψ has the external $\tilde{C}$ charge equal $2M$, the covariant derivatives $\bar{\mathcal{D}}^1$, $\mathcal{D}_z$, $\mathcal{D}_{\bar{z}}$ have, respectively, the $\tilde{C}$ charges $+1, +1$

[12] One can assign to Ψ also a non-zero external charge associated with the outer automorphisms U(1) generator J the differential part of which is given in (2.30). However, this U(1) has no actual implications in the considered model.

and -1, while $\mathcal{D}^2, \bar{\mathcal{D}}_2$ are $\tilde{C}$-neutral. This $\tilde{C}$ assignment should be kept in mind while checking the relations (3.66), (3.67). The standard (non-covariant) commutation relations (without taking account of the non-trivial $\tilde{C}$ connection terms in $\mathcal{D}_2, \bar{\mathcal{D}}^2$) can be easily derived from the above covariant ones.

Representing the covariant derivatives $\bar{\mathcal{D}}^i$ on Ψ (i.e. with $\tilde{C} = 2M$) by

$$\bar{\mathcal{D}}^1 = K_1^{\frac{1}{2}} \bar{\varphi}^1 , \quad \bar{\mathcal{D}}^2 = \bar{\varphi}^2 + \bar{z}\bar{\varphi}^1 , \tag{3.68}$$

it is easy to see that the physical state conditions (3.32) are equivalent to

$$\bar{\mathcal{D}}^i \Psi = 0 , \tag{3.69}$$

which is the standard covariant form of the chirality conditions. The prefactors in the solution (3.34) serve to eliminate the connection terms in $\bar{\mathcal{D}}^i$ when the latter act on the reduced wave function Φ. After that, the conditions (3.69) are solved by passing to the chiral basis $(z, \bar{z}_{sh})$. The derivative $\mathcal{D}_{\bar{z}}$ also becomes short on $\Phi(z, \bar{z}_{sh}, \xi^i)$: $\mathcal{D}_{\bar{z}} \rightarrow \tilde{\mathcal{D}}_{\bar{z}} = K_1^{\frac{1}{2}} \partial_{\bar{z}_{sh}}$. Thanks to the commutation relations (3.66), it is then consistent to impose the additional analyticity constraint on the ground state $\Phi(z, \bar{z}_{sh}, \xi^i)$, viz. $\tilde{\mathcal{D}}_{\bar{z}} \Phi = 0 \rightarrow \Phi = \Phi_0(z, \xi^i)$.

When dealing with the eigenvalue problem of the Hamiltonian in the previous subsection, we worked with the operators $\nabla_z , \nabla_{\bar{z}}$, which can be treated as a type of creation and annihilation operator (see (3.30)). Using the covariant derivatives $\mathcal{D}_z, \mathcal{D}_{\bar{z}}$, eq. (3.64), the analogy with the quantum oscillator becomes literal, because their commutator equals a constant and the Hamiltonian can be rewritten in the standard oscillator form:

$$\hat{H} = -\mathcal{D}_z \mathcal{D}_{\bar{z}} + \kappa . \tag{3.70}$$

The eigenvector for the Landau level N can be rewritten as

$$\Psi^{(N)} = (\mathcal{D}_z)^N K_1^{M - \frac{N}{2}} e^{-\kappa K_2} \Phi_0^{(N)} \left(z, \xi^i\right) . \tag{3.71}$$

The corresponding ground state reduced wave function $\Phi_0^{(N)}$ has $\tilde{C} = 2M - N$, while the whole $\Psi^{(N)}$ has $\tilde{C} = 2M$, since each $\mathcal{D}_z$ adds $\tilde{C} = 1$. The formula (3.43) for the energy levels can be equivalently derived using the commutation relations (3.64). Note that the Hamiltonian commutes with the chirality constraints (3.69) in a weak sense, $[\hat{H}, \bar{\mathcal{D}}^2] \sim \bar{\varphi}^1$.

4. Summary

In previous papers we solved the Landau problem for a particle on the supersphere SU(2|1/ U(1|1) and the superflag SU(2|1)/[U(1) × U(1)]. The latter coset superspace allows two WZ terms, and hence a family of Landau models, for fixed magnetic field, parametrized by the coefficient M of a "fermionic Wess-Zumino" term. The equivalence of the $M = 0$ model with the supersphere Landau model was implicit in these results, but not explained by them. In this paper we have reconsidered these models in the planar limit.

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The supersphere model becomes the "superplane" Landau model for a particle on $\mathbb{C}^{(1|1)}$; this is a model with a quadratic Lagrangian that is the sum of the standard Landau model with a four-state "fermionic Landau model". The latter has just two Landau levels, each spanned by two states, with the excited states having negative norm. This provides a simple explanation for the negative norm states, or "ghosts", in all but the lowest Landau level of the supersphere model, and it shows clearly that ghosts arise as a result of second-order fermion kinetic terms.

The planar limit of the superflag model yields a model that we have called the "planar superflag" Landau model. It is an extension of the superplane to include interactions with an additional Goldstino variable. For $M = 0$ this variable is auxiliary and the superplane model is recovered on eliminating it; this explains the equivalence between the superplane and $M = 0$ superflag models. The motivation for considering the $M > 0$ superflag model (planar or spherical) is that the second-order fermion kinetic terms responsible for ghosts are "suppressed" in the sense that the coefficient becomes nilpotent. As a result, the ghosts are not eliminated entirely but just banished to the higher Landau levels. Specifically, the Nth level is ghost-free if and only if $N \leq 2M$.

Another curious, and related, feature of the $M > 0$ planar superflag models is that the second class fermionic constraints (which are standard in models with anticommuting variables) cease to be entirely second-class on a fixed-energy subspace of the phase space, thus implying the presence of a gauge-invariance on this energy surface. In fact, when restricted to this exceptional energy the planar superflag Landau model becomes a type of time-reparametrization invariant superparticle model with a "hidden" fermionic gauge invariance. However, this gauge invariance has an effect on the quantum theory only when the exceptional energy surface is one of the Landau levels, and this happens only when $2M$ is an integer. In this case, the fermionic gauge invariance leads to short supermultiplets for the states at the $(2M + 1)$th Landau level, this being the lowest Landau level for $M = 0$. The short supermultiplets are exactly as expected from our previous results for the supersphere and superflag Landau models.

Although the super-Landau models analysed here have ghosts, it is possible to consistently truncate to a ghost free theory. One could throw out just the ghosts, but this would break the SU(2|1) symmetry that was the rationale for the construction of these models. Instead, one can throw out all Landau levels that contain ghosts. For $M = 0$ this is equivalent to keeping only the lowest Landau level, which defines the non-(anti)commutative complex superplane that results from taking the planar limit of the fuzzy supersphere. Our $M > 0$ planar superflag models, truncated to the first $2[M] + 1$ levels, can be considered as generalizations of this construction to allow for a finite set of higher Landau levels. As the Hilbert space still has finite dimension, the quantum theory defines a fuzzy version of the supermanifold obtained from the planar limit of the superflag.

Note added. After submission to the archives, we learnt of a paper of Hasebe [10] in which a planar super-Landau model is obtained as the planar limit of a Landau model for a particle on the coset superspace OSp(1|2)/ U(1). This "supersphere" has real dimension (2|4), and is therefore "non-minimal" in comparison to the supersphere defined here as

$CP^{(1|1)}$, but it can be viewed as a superspace of real dimension (2|2) with the help of a "pseudoconjugation" operation that squares to -1 when acting on spinors. This leads to a planar super-Landau model that is superficially equivalent to the superplane Landau model discussed here. However the wave superfunction of the present paper and that of [10], are functions on superspaces with different involutions. The relation between these two superspaces is not known to us; our wave superfunctions are essentially ordinary superfields. We stress that our interest here is in quantum models carrying unitary representations of IU(1|1); the natural norm chosen here is the unique norm generating invariant theories. We believe that the norm of [10] is a "bi-orthogonal" norm [11] (see also [12]), and we plan to return to this point in a future work with T. Curtright.

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References

[1] L.D. Landau, *Diamagnetismus der Metalle*, *Z. Phys.* **64** (1930) 629.

[2] F.D.M. Haldane, *Fractional quantization of the Hall effect: a hierarchy of incompressible quantum fluid states*, *Phys. Rev. Lett.* **51** (1983) 605.

[3] J. Madore, *The fuzzy sphere*, *Class. and Quant. Grav.* **9** (1992) 69.

[4] E. Ivanov, L. Mezincescu and P.K. Townsend, *Fuzzy $CP(N|M)$ as a quantum superspace*, in *Symmetries in gravity and field theory*, V. Aldaya, J.M. Cerveró and P. García eds., Ediciones Universidad de Salamanca, 2004, pp. 385–408 [hep-th/0311159].

[5] E. Ivanov, L. Mezincescu and P.K. Townsend, *A super-flag Landau model*, in *From fields to strings: circumnavigating theoretical physics*, M. Shifman, A. Vainshtein and J. Wheater eds., World Scientific, Singapore 2004 [hep-th/0404108].

[6] M. Bañados, L.J. Garay and M. Henneaux, *The dynamical structure of higher dimensional Chern-Simons theory*, *Nucl. Phys.* **B 476** (1996) 611 [hep-th/9605159].

[7] E. Ivanov, L. Mezincescu, A. Pashnev and P.K. Townsend, *Odd coset quantum mechanics*, *Phys. Lett.* **B 566** (2003) 175 [hep-th/0301241].

[8] M. Hatsuda, S. Iso and H. Umetsu, *Noncommutative superspace, supermatrix and lowest Landau level*, *Nucl. Phys.* **B 671** (2003) 217 [hep-th/0306251].

[9] E.A. Ivanov and V.I. Ogievetsky, *The inverse Higgs phenomenon in nonlinear realizations*, *Teor. Mat. Fiz.* **25** (1975) 164.

[10] K. Hasebe, *Supersymmetric extension of noncommutative spaces, berry phases and quantum Hall effects*, hep-th/0503162.

[11] C.L. Bender, *Introduction to $\mathcal{PT}$-symmetric quantum theory*, quant-ph/0501052.

[12] T. Curtright and L. Mezincescu, *Biorthogonal quantum systems*, quant-ph/0507015.